Int. J. Solids Structures Vol. 19, No. 7, pp. 643–652, 1983
Printed in Great Britain.

0020–7683/83 $3.00+.00

THERMO-ELASTIC/PLASTIC BEHAVIOUR OF A STRAIN-HARDENING THICK-WALLED SPHERE

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(*Received* 2 *August* 1982)

Abstract—An exact solution is presented for the thermo-elastic/plastic problem of a thick-walled sphere subjected to a radial temperature gradient. Material behaviour is described by the von-Mises flow rule in conjunction with a modified Ramberg–Osgood stress–strain characteristic. The problem is formulated within the framework of small strain plasticity. Contact is made with earlier studies for elastic/perfectly-plastic and pure power-hardening materials. Also given are a few basic results for elastic/linear-hardening materials.

INTRODUCTION

The spherical symmetric behaviour of an elastic/perfectly-plastic thick-walled sphere, subjected to a radial temperature gradient, has been studied analytically in [1, 2]. Recently, [3], an exact solution has been given for the same problem using a pure power-hardening law. These studies treated the steady-state, time independent, thermo-elastic/plastic behaviour. The more difficult transient thermal problem has been investigated numerically in [4, 5].

Here we investigate the steady-state problem for an elastic/plastic material, assuming a modified Ramberg–Osgood stress–strain relation. The uniaxial stress–strain curve is commonly written as

$$\epsilon = \Sigma + \epsilon_p \tag{1}$$

where ϵ is the total strain, the nondimensional effective stress Σ is identified here with the uniaxial stress σ divided by the elastic modulus E, and ϵ_p is the total plastic strain and a known function of Σ. In the present paper we borrow an idea from Budiansky[6] and assume that

$$\epsilon_P = 0 \quad \text{for} \quad \Sigma \leqslant \Sigma_Y, \quad \text{with} \quad \Sigma_Y = \frac{Y}{E} \tag{2a}$$

$$\epsilon_P = \Sigma\left[\left(\frac{\Sigma}{\Sigma_Y}\right)^{n-1} - 1\right] - (1-2\nu)(\Sigma - \Sigma_Y) \quad \text{for} \quad \Sigma \geqslant \Sigma_Y \tag{2b}$$

where Y is the yield stress, n is the hardening parameter and ν is Poisson's ratio. For an incompressible material $\nu = 1/2$ and relation (2b) becomes identical with the one employed in [6] for investigating a stress concentration problem. These modifications of the Ramberg–Osgood relation offer the advantage of deriving exact solutions for the respective problems. Comparing modification (2b) with the original relation suggested by Budiansky in [6], we note that the ratio of the respective total strains (1) is $1-\eta$ where

$$\eta = (1-2\nu)\left(1 - \frac{\Sigma_Y}{\Sigma}\right)\left(\frac{\Sigma_Y}{\Sigma}\right)^{n-1}, \quad \Sigma \geqslant \Sigma_Y. \tag{3}$$

It can now easily be shown that $\eta \ll 1$. The highest value of η is attained at the stress level $\Sigma = (n/(n-1))\Sigma_Y$. With $\nu = 1/3$, for example, we have η_{max} $(n=3) = 0.049$, η_{max} $(5) = 0.027$, η_{max} $(10) = 0.013$ and η_{max} $(20) = 0.0063$. For higher values of the hardening parameter n, the maximum value of η approaches the asymptote $(1-2\nu)/en$. Thus, the present modification of the Ramberg–Osgood relation remains in close agreement with the modification used in [6].

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More important, the uniaxial stress–strain curve of many metals can be closely approximated by relations (2a, b), thus facilitating the analysis of our problem.

The basic equations are set in the next section within the framework of small strain plasticity. The radial temperature gradient is assumed to be given by the steady state profile that corresponds to outward radial flow of heat. Subsequently, we analyze the various stages of the loading path. The overall picture is similar to the one presented in [1, 2] for an elastic/perfectly-plastic material. Yielding begins at the inner surface and a second plastic zone may occur, with further increase in the temperature gradient. The solution centers on determining the exact location of the elastic/plastic interfaces as a function of geometry, material hardening and temperature gradient. Contact is made with the earlier results presented in [1–3].

Finally, we give a brief discussion of this thermo-elastic/plastic problem for an elastic/linear-hardening material. Relation (2b) is now replaced by

$$\epsilon_p = \frac{1-h}{h}(\Sigma - \Sigma_Y) \quad \text{for} \quad \Sigma \geqslant \Sigma_Y, \quad \text{with} \quad h = \frac{E_T}{E} \tag{4}$$

where E_T is the constant tangent modulus.

While the analysis for that material is simpler in comparison with material (2b), it has to be noted that relation (4) provides a less realistic approximation to the actual uniaxial stress–strain curve.

PROBLEM FORMULATION

A thick walled sphere ($a \leqslant r \leqslant b$) is slowly heated by raising monotonously the temperature difference between the boundaries. Observing the spherical symmetry of the field and assuming (as will be verified later) continuous plastic loading, we can write the constitutive relations as

$$\epsilon_r = \Sigma_r - 2\nu\Sigma_\theta + \alpha T - m\epsilon_P \tag{5}$$

$$\epsilon_\theta = (1-\nu)\Sigma_\theta - \nu\Sigma_r + \alpha T + \frac{1}{2} m\epsilon_P \tag{6}$$

where, with the usual notation, ϵ_r and ϵ_θ are the strain components, Σ_r and Σ_θ are the stress components (nondimensionalized with respect to E), α is the coefficient of thermal expansion, and T the temperature change. The effective stress is here conveniently written as

$$\Sigma = -m(\Sigma_r - \Sigma_\theta) \quad \text{with} \quad m = -\operatorname{sgn}(\Sigma_r - \Sigma_\theta). \tag{7}$$

The strains are related to the radial displacement w through

$$\epsilon_r = \frac{\mathrm{d}w}{\mathrm{d}r}, \quad \epsilon_\theta = \frac{w}{r} \tag{8}$$

which, when combined, result in the compatibility equation

$$\frac{\mathrm{d}\epsilon_\theta}{\mathrm{d}\rho} - \frac{\epsilon_r - \epsilon_\theta}{\rho} = 0 \tag{9}$$

where $\rho = r/a$ is the nondimensional radial coordinate ($1 \leqslant \rho \leqslant \beta$ where $\beta = b/a$ is the radii ratio).

Finally, we have the single equilibrium equation which, in view of (7), may be put in the form

$$\frac{\mathrm{d}\Sigma_r}{\mathrm{d}\rho} = 2m\frac{\Sigma}{\rho}. \tag{10}$$

Inserting now (5) and (6) in (9), noting (7) and using (10), gives the basic differential equation

$$\frac{d}{d\rho}[2(1-\nu)\Sigma+\epsilon_p]+\frac{3}{\rho}[2(1-\nu)\Sigma+\epsilon_P]=-2m\alpha\frac{dT}{d\rho} \tag{11}$$

or, after integration,

$$2(1-\nu)\Sigma+\epsilon_P=-\frac{2m\alpha}{\rho^3}\int\rho^3\left(\frac{dT}{d\rho}\right)d\rho+\text{constant.} \tag{12}$$

In this study we assume that the radial temperature profile is given by the steady state distribution, with the corresponding gradient

$$\frac{dT}{d\rho}=-\Delta T\frac{k}{\rho^2}\quad\text{with}\quad k=\frac{\beta}{\beta-1},\quad \beta=\frac{b}{a} \tag{13}$$

and ΔT stands for the positive temperature difference between the inner and outer surfaces.

Substituting (13) in (12), integrating and arranging, we obtain

$$2(1-\nu)\Sigma+\epsilon_P=m\theta k\left(\frac{1}{\rho}-\frac{C}{\rho^3}\right),\quad \theta=\alpha\Delta T \tag{14}$$

where C is a constant. In passing, however, we note that the subsequent analysis is not restricted to the steady state temperature distribution.

(Relation (14) clarifies the advantage of using representation (2b); inserting the latter in (14) leads to an explicit expression for the radial dependence of the effective stress. It is worth mentioning, however, that (2b) is not the only possible modification, of the Ramberg–Osgood relation, with that useful property. Just to give an example, consider the two parameter family

$$\epsilon_P=K\left[\left(\frac{\Sigma}{\Sigma_Y}\right)^n-1\right]-2(1-\nu)(\Sigma-\Sigma_Y)\quad\text{for}\quad \Sigma\geqslant\Sigma_Y$$

where n, K are material constants. The tangent modulus at the yield point $\Sigma=\Sigma_Y$ can be made continuous with the choice $K=2(1-\nu)\Sigma_Y/n$.)

Prior to the occurence of plastic yielding, the whole sphere is in a purely elastic state with $\epsilon_P\equiv 0$. This is of course a well known problem and we shall just recapitulate the basic results for future use. Relation (14) becomes here

$$\Sigma=\frac{m\theta k}{2(1-\nu)}\left(\frac{1}{\rho}-\frac{C}{\rho^3}\right). \tag{15}$$

Inserting (15) in (10) and integrating over ρ, with the initial condition $\Sigma_r=0$ at $\rho=1$, gives the radial stress component

$$\Sigma_r=\frac{\theta k}{1-\nu}\left[1-\frac{1}{\rho}-\frac{C}{3}\left(1-\frac{1}{\rho^3}\right)\right]. \tag{16}$$

The value of constant C follows from the condition that the outer surface ($\rho=\beta$) is stress free ($\Sigma_r=0$), namely

$$C=\frac{3\beta^2}{\beta^2+\beta+1} \tag{17}$$

The radial stress (16) is now completely determined. The circumferential stress component is then obtained from (15) to (16) with the aid of the first of (7), and the radial displacement follows from (6) and the second of (8).

Combining (15) with (17) we find, [1–2], that the effective stress attains its highest value at the inner surface. Yielding will therefore begin at $\rho = 1$ when $\Sigma = \Sigma_Y$, i.e. when the temperature difference is

$$\theta^* = \frac{\beta^2 + \beta + 1}{\beta(2\beta + 1)} \tag{18}$$

where

$$\theta^* = \frac{\theta}{2(1-\nu)\Sigma_Y} = \frac{\alpha \Delta T E}{2(1-\nu)Y}. \tag{19}$$

The value of m used in deriving (18) was $m = -1$. A helpful notion in this context is that of the "hydrostatic surface" $\rho = \rho_0$ where all stress components are equal. Obviously, the effective stress (15) vanishes at ρ_0 so that in the complete elastic state

$$\rho_0^2 = \frac{3\beta^2}{\beta^2 + \beta + 1}. \tag{20}$$

It is now a matter of ease to verify that $m = -1$ for $1 \leqslant \rho \leqslant \rho_0$ and $m = 1$ for $\rho_0 \leqslant \rho \leqslant \beta$, since the effective stress is always positive.

THE FIRST PLASTIC ZONE

The onset of plastic yielding at the inner surface is followed, as the temperature difference increases beyond (18), by the growth of a plastic zone spanning over the range $1 \leqslant \rho \leqslant \rho_1$ where ρ_1 denotes the elastic/plastic interface. The basic relation within that plastic zone is obtained by substituting the effective stress (2b) in (14), with $m = -1$, resulting in

$$\Sigma\left(\frac{\Sigma}{\Sigma_Y}\right)^{n-1} + (1-2\nu)\Sigma_Y = \theta k\left(\frac{C}{\rho^3} - \frac{1}{\rho}\right). \tag{21}$$

However, at the interface $\rho = \rho_1$ we have $\Sigma = \Sigma_Y$, and so

$$C = \rho_1^2\left(1 + \frac{\rho_1}{\tau}\right) \tag{22}$$

where

$$\tau = k\theta^* = \left(\frac{\beta}{\beta - 1}\right)\frac{\alpha \Delta T E}{2(1-\nu)Y}. \tag{23}$$

The effective stress within the plastic zone follows now from (21) to (22) as

$$\Sigma = \Sigma_Y[2(1-\nu)]^{1/n}\left[\frac{\rho_1^2(\rho_1 + \tau)}{\rho^3} - \frac{\tau}{\rho} - \frac{1-2\nu}{2-2\nu}\right]^{1/n}. \tag{24}$$

Inserting (24) in the equilibrium equation (10), with $m = -1$, and integrating over ρ, with the initial condition $\Sigma_r = 0$ at $\rho = 1$, gives

$$\Sigma_r = -2\Sigma_Y[2(1-\nu)]^{1/n}\int_1^\rho \left[\frac{\rho_1^2(\rho_1 + \tau)}{\rho^3} - \frac{\tau}{\rho} - \frac{1-2\nu}{2-2\nu}\right]^{1/n}\frac{d\rho}{\rho}. \tag{25}$$

Within the elastic zone, $\rho_1 \leqslant \rho \leqslant \beta$, we have again relation (15) with the condition that at the interface $\Sigma = \Sigma_Y$, with $m = -1$. Constant C is therefore again given by (22) and the effective stress becomes

$$\Sigma = m\Sigma_Y \frac{(\rho^2-\rho_1^2)\tau-\rho_1^3}{\rho^3} \quad \rho_1 \leq \rho \leq \beta \tag{26}$$

where $m = -1$ for $\rho_1 \leq \rho \leq \rho_0$, and $m = 1$ for $\rho_0 \leq \rho \leq \beta$. The hydrostatic surface is here given by

$$\rho_0^2 = \rho_1^2\left(1+\frac{\rho_1}{\tau}\right). \tag{27}$$

Substituting (26) in (10) and integrating over ρ, with the condition that $\Sigma_r = 0$ at $\rho = \beta$, we get

$$\Sigma_r = -2\Sigma_Y\left(\frac{\beta-\rho}{\beta\rho}\right)\left[\tau - \frac{\rho_1^2(\rho_1+\tau)(\rho^2+\beta\rho+\beta^2)}{3\beta^2\rho^2}\right] \quad \rho_1 \leq \rho \leq \beta. \tag{28}$$

Now, continuity of radial stress at the elastic/plastic interface requires that expressions (25) and (28) should agree at $\rho = \rho_1$. This condition leads to the transcendental equation

$$[2(1-\nu)]^{1/n}\int_1^{\rho_1}\left[\frac{\rho_1^2(\rho_1+\tau)}{\rho^3}-\frac{\tau}{\rho}-\frac{1-2\nu}{2-2\nu}\right]^{1/n}\frac{d\rho}{\rho} = \left(\frac{\beta-\rho_1}{\beta\rho_1}\right)\left[\tau-\frac{(\rho_1+\tau)(\rho_1^2+\beta\rho_1+\beta^2)}{3\beta^2}\right] \tag{29}$$

which determines the location of the interface ρ_1 as a function of geometry (β), temperature difference (τ) and material properties (n, ν). When $n \to \infty$ equation (29) reduces to the one derived in [1, 2], for an elastic/perfectly-plastic material, where the l.h.s. of (29) is replaced by $\ln \rho_1$.

Once the value of ρ_1 has been determined, from (29), the entire stress field within the sphere is readily available. For the plastic phase we have (24) and (25) together with the first of (7), while for the elastic phase we have again the first of (7), (26) and (28). The corresponding expressions for the radial displacement follow then without difficulty.

The relations derived hitherto, in this section, hold as long as no second plastic zone arises. For sufficiently "thick" shells, second yield will occur within the elastic zone at the radius $\rho = \rho_2$ where the effective stress (26) has a maximum. Differentiating (26) we find that this happens at

$$\rho_2 = \sqrt{(3)}\rho_1\sqrt{\left(1+\frac{\rho_1}{\tau}\right)} \tag{30}$$

where τ and ρ_1 are here evaluated at the onset of the second plastic zone. Observing now that $\Sigma = \Sigma_Y$ at $\rho = \rho_2$ we obtain from (26), with $m = 1$, a second relation for (ρ_2, ρ_1, τ). Combining the latter with (30) we find that

$$\rho_2 = \frac{2}{3}\tau, \quad \rho_1 = \frac{1}{3}\tau \tag{31}$$

at the initiation of a second plastic zone in "thick" shells. Note that $\rho_2 = 2\rho_1$ and that the corresponding value of the hydrostatic surface is $\rho_0 = (2/\sqrt{3})\rho_1 = (2/3\sqrt{3})\tau$. The critical value of τ is obtained, from (29) and the second of (31), as the solution of the equation

$$[2(1-\nu)]^{1/n}\int_1^{(1/3)\tau}\left[\frac{4\tau^3}{27\rho^3}-\frac{\tau}{\rho}-\frac{1-2\nu}{2-2\nu}\right]^{1/n}\frac{d\rho}{\rho} = \left(\frac{3\beta-\tau}{81\beta^3}\right)(45\beta^2-12\beta\tau-4\tau^2) \tag{32}$$

When $n \to \infty$ the l.h.s. of (32) reduces to $\ln(\tau/3)$ in agreement with the result in [2] for an elastic/perfectly-plastic material.

On the other hand, if the shell is sufficiently "thin" then the second yield zone will always begin at the outer surface $\rho = \beta$. The limiting value of β, which separates "thick" shells from "thin" shells, is obtained from the first of (31), with $\rho_2 \equiv \beta = (2/3)\tau$, together with (32). Thus

$$(1-\nu)^{1/n}\int_1^{(1/2)\beta}\left(\frac{\beta^3}{\rho^3}-\frac{3\beta}{\rho}-\frac{1-2\nu}{1-\nu}\right)^{1/n}\frac{d\rho}{\rho}=\frac{1}{3}. \tag{33}$$

The solution of (33) is shown in Fig. 1 as a function of the hardening parameter n and for two different values of ν. It may be seen that Poisson's ratio has little effect on the limiting value of β. For large values of n the limiting value of β approaches asymptotically the limiting value for an elastic/perfectly-plastic material. Equation (33) is then simply in $(1/2)\beta = 1/3$ with the solution, [1]–[2], $\beta = 2.791$. The approach towards that asymptote is shown in Fig. 1 and we may conclude that the hardening parameter does not influence much the limiting value of β.

For "thin" shells, with radii ratios below the limiting value given by (33), we get from (26) that at the onset of the second yield zone

$$\tau = \beta + \frac{\rho_1^2}{\beta-\rho_1} \tag{34}$$

where the values of $\Sigma = \Sigma_Y$, $\rho = \beta$ and $m = 1$ have been used. Substituting (34) in (29) we arrive at a single equation for the location of the elastic/plastic interface ρ_1, when the outer surface just reaches the yield point, namely

$$[2(1-\nu)]^{1/n}\int_1^{\rho_1}\left[\left(\frac{\beta^2\rho_1^2}{\beta-\rho_1}\right)\frac{1}{\rho^3}-\left(\beta+\frac{\rho_1^2}{\beta-\rho_1}\right)\frac{1}{\rho}-\frac{1-2\nu}{2-2\nu}\right]^{1/n}\frac{d\rho}{\rho}=\frac{2(\beta-\rho_1)^2}{3\beta\rho_1}. \tag{35}$$

The solution of this equation provides, via (34), the temperature difference that corresponds to the initiation of a second plastic zone at the outer surface.

Figure 2 displays the dependence of the temperature difference θ^*, as obtained from (32) for "thick" shell, and from (35) to (34) for "thin" shells, on β and material properties. Note again that with $n \to \infty$ the l.h.s. of (35) is simply $\ln \rho_1$ in agreement with [1, 2]. Also shown in Fig. 2 is the temperature difference (18) required for first plastic yielding to occur at the inner surface.

THE SECOND PLASTIC ZONE FOR "THIN" SHELLS

If the temperature difference is raised even further, after the occurrence of second yield, then a second plastic zone will develop and increase in size as ΔT increases. In this section we shall consider the more practical case of "thin" shells where the second plastic zone starts at the outer surface. There is however no difficulty in extending the following analysis to "thick" shells where second yield occurs within the shell.

The shell is now divided into three different zones; the first plastic zone $1 \leqslant \rho \leqslant \rho_1$, the elastic zone $\rho_1 \leqslant \rho \leqslant \rho_2$, and the second plastic zone $\rho_2 \leqslant \rho \leqslant \beta$. The problem lies in finding the location of the elastic/plastic interfaces (ρ_1, ρ_2) as a function of geometry, material properties and temperature difference.

The results derived earlier, (21)–(25), for a single plastic zone apply here as well for the first plastic zone. For the elastic zone we take (15) and implement the two yield conditions $\Sigma = \Sigma_Y$ at

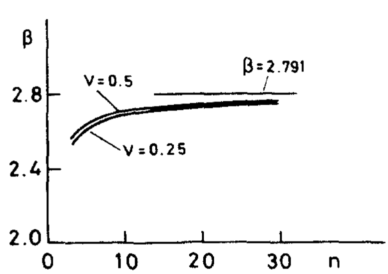

Fig. 1. The limiting value of the radii ratio β as a function of the hardening parameter n. The asymptotic value $\beta = 2.791$ is the exact solution for an elastic/perfectly-plastic material.

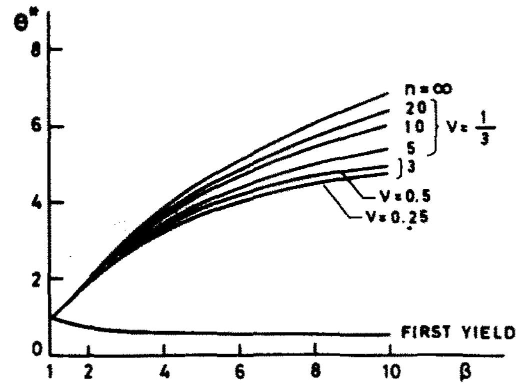

Fig. 2. Temperature difference required for the onset of a second plastic zone. The lower curve shows the temperature difference when yielding first begins at the inner surface.

$\rho = \rho_1$ (with $m = -1$) and at $\rho = \rho_2$ (with $m = 1$). This gives the two relations

$$C = \frac{(\rho_2\rho_1)^2}{\rho_2^2 - \rho_1\rho_2 + \rho_2^2} \equiv \rho_0^2 \tag{36}$$

$$\tau = \frac{\rho_2^2 - \rho_1\rho_2 + \rho_1^2}{\rho_2 - \rho_1}. \tag{37}$$

Note that the location of the hydrostatic surface ($\Sigma = 0$) is now given by (36).

The distribution of the radial stress within the elastic zone follows from (10) and (15), and is governed by the differential relation

$$d\Sigma_r = 2\Sigma_Y\tau\left(\frac{1}{\rho^2} - \frac{C}{\rho^4}\right) d\rho \tag{38}$$

where C is given by (36).

Within the second plastic zone we get from (14), with $m = 1$, and from (2b) that

$$\Sigma\left(\frac{\Sigma}{\Sigma_Y}\right)^{n-1} + (1-2\nu)\Sigma_Y = \theta k\left(\frac{1}{\rho} - \frac{C}{\rho^3}\right). \tag{39}$$

At the interface $\rho = \rho_2$ we have $\Sigma = \Sigma_Y$, which determines C as

$$C = \rho_2^2\left(1 - \frac{\rho_2}{\tau}\right). \tag{40}$$

Combining (39) with (40) gives the expression for the effective stress, within the second plastic zone, viz

$$\Sigma = \Sigma_Y[2(1-\nu)]^{1/n}\left[\frac{\rho_2^2(\rho_2 - \tau)}{\rho^3} + \frac{\tau}{\rho} - \frac{1-2\nu}{2-2\nu}\right]^{1/n}. \tag{41}$$

Inserting now (41) in (10), with $m = 1$, and integrating over ρ with the initial condition $\Sigma_r = 0$ at $\rho = \beta$, we find that

$$\Sigma_r = -2\Sigma_Y[2(1-\nu)]^{1/n}\int_\rho^\beta \left[\frac{\rho_2^2(\rho_2 - \tau)}{\rho^3} + \frac{\tau}{\rho} - \frac{1-2\nu}{2-2\nu}\right]^{1/n}\frac{d\rho}{\rho}. \tag{42}$$

Finally, we integrate (38) over the elastic zone and substitute the boundary values, at the interfaces ρ_1, ρ_2, as obtained from (25) and (42) respectively. The result can be put in the form

$$\int_1^{\rho_1}\left[\frac{\rho_1^2(\rho_1+\tau)}{\rho^3}-\frac{\tau}{\rho}-\frac{1-2\nu}{2-2\nu}\right]^{1/n}\frac{d\rho}{\rho}-\int_{\rho_2}^{\beta}\left[\frac{\rho_2^2(\rho_2-\tau)}{\rho^3}+\frac{\tau}{\rho}-\frac{1-2\nu}{2-2\nu}\right]^{1/n}\frac{d\rho}{\rho}=\frac{2}{3}[2(1-\nu)]^{-1/n}\frac{(\rho_2-\rho_1)^2}{\rho_1\rho_2}. \tag{43}$$

Equation (43) together with relation (37) may now be solved quite easily, using a standard numerical scheme, for the unknowns ρ_1, ρ_2 as a function of τ, β, n, ν. The associated expressions for the stress components and the radial displacement, within the entire sphere, follow then almost immediately.

With $n \to \infty$ eqn (43) takes the simple form

$$\ln\frac{\rho_1\rho_2}{\beta}=\frac{2(\rho_2-\rho_1)^2}{3\rho_1\rho_2} \tag{44}$$

which, together with (37), agree with the equivalent relations derived in [1] for an elastic/perfectly-plastic material.

The asymptotic behaviour of the shell, as τ becomes very large, is given by the exact solution for a pure power-hardening material, [3]. Indeed, when $\tau \to \infty$ we have that $\rho_1 \to \rho_2 \to \rho_0$, where the equation for ρ_0 follows from (43) as

$$\int_1^{\rho_0}\left(\frac{\rho_0^2}{\rho^3}-\frac{1}{\rho}\right)^{1/n}\frac{d\rho}{\rho}-\int_{\rho_0}^{\beta}\left(\frac{1}{\rho}-\frac{\rho_0^2}{\rho^3}\right)^{1/n}\frac{d\rho}{\rho}=0. \tag{45}$$

Equation (45) agrees, except a notational change, with the result derived earlier in [3]. That reference contains also a chart presenting the solution of (45) as a function of β and n.

Figure 3 shows the movement of the elastic/plastic interfaces, and the hydrostatic surface, as the temperature difference increases, for the particular geometry $\beta = 2$ with $n = 3$ ($\nu = 1/3$) and with $n = \infty$. The curves for the case $n = 3$ were computed, according to the various stages of the loading history, from (20), (29) and (27), and from (43) with (36) and (37). For the elastic/perfectly-plastic material ($n = \infty$) we have used the simpler relations that do not involve numerical integration. Note that as θ^* increases both interfaces approach the hydrostatic surface given by (45).

A similar analysis for "thick" shells leads to two equations similar to (43), together with a relation like (37), for the three elastic /plastic interfaces. But even in that case the numerical solution does not present a great difficulty.

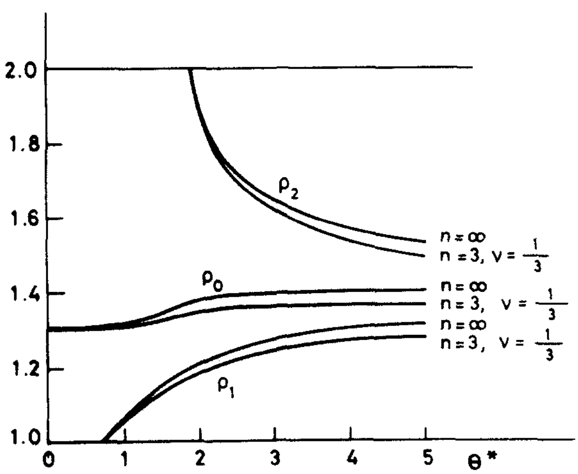

Fig. 3. Location of the elastic/plastic interfaces and the hydrostatic surface, as a function of temperature difference, for $\beta = 2$, with $n = 3(\nu = 1/3)$ and $n = \infty$. Note that both interfaces approach the hydrostatic surface as θ^* increases.

A FEW BASIC RESULTS FOR ELASTIC/LINEAR-HARDENING MATERIALS

In the last section of this paper we present some useful relations for the elastic/linear-hardening model (4). Within the first plastic zone we can write the analogue of (21) as

$$\left[2(1-\nu)+\frac{1-h}{h}\right]\Sigma-\frac{1-h}{h}\Sigma_Y=\theta k\left(\frac{C}{\rho^3}-\frac{1}{\rho}\right) \tag{46}$$

where C is again given by (22). The expression for the effective stress, for $1\leq\rho\leq\rho_1$, is therefore

$$\Sigma=\Sigma_Y\left\{\frac{1}{1+H}+\frac{H}{1+H}\left[\frac{\rho_1^2(\rho_1+\tau)}{\rho^3}-\frac{\tau}{\rho}\right]\right\} \tag{47}$$

where H is the apparent tangent modulus defined by

$$H=2(1-\nu)\frac{h}{1-h}. \tag{48}$$

Inserting (47) in (10) with $m=-1$, and integrating we get the distribution of radial stress component within the first plastic zone. The location of the elastic/plastic interface ρ_1 follows upon equating the radial stress on both sides of the interface. This leads to the transcendental equation

$$\frac{1}{1+H}\left[\ln\rho_1+H\frac{(\rho_1+\tau)(\rho_1^3-1)-3\tau(\rho_1-1)}{3\rho_1}\right]=\left(\frac{\beta-\rho_1}{\beta\rho_1}\right)\left[\tau-\frac{(\rho_1+\tau)(\rho_1^2+\beta\rho_1+\beta^2)}{3\beta^2}\right] \tag{49}$$

which may be compared with (29).

The onset of a second plastic zone for "thick" shells is again determined by relations (31). Combining the second of (31) with (49) we obtain the equation for the critical temperature difference, that corresponds to the initiation of the second plastic zone within the shell, namely

$$\frac{1}{1+H}\left[\ln\frac{\tau}{3}+H\left(\frac{4\tau^3}{81}-\tau+\frac{5}{3}\right)\right]=\left(\frac{3\beta-\tau}{81\beta^3}\right)(45\beta^2-12\beta\tau-4\tau^2). \tag{50}$$

For "thin" shells, where second yielding begins at the outer surface, we combine relation (34) with (49) in order to get the equation

$$\frac{1}{1+H}\left\{\ln\rho_1+H\frac{\rho_1-1}{3\rho_1(\beta-\rho_1)}[\beta^2(\rho_1+2)(\rho_1-1)+3\rho_1(\beta-\rho_1)]\right\}=\frac{2(\beta-\rho_1)^2}{3\beta\rho_1} \tag{51}$$

for the location of the first elastic/plastic interface when yielding just occurs at $\rho=\beta$. The solution of (51) provides, via (34), the corresponding temperature difference.

The limiting value of β, below which the shell is regarded as "thin", follows either from (50), or from (51) together with (34), when $\beta=(2/3)\tau=2\rho_1$. The final equation for the limiting value of β can be written in the form

$$\ln\frac{1}{2}\beta-\frac{1}{3}+H\left(\frac{1}{6}\beta^3-\frac{3}{2}\beta+\frac{4}{3}\right)=0. \tag{52}$$

Increasing the hardening parameter causes a decrease in the limiting radii ratio; starting with $\beta(H=0)=2.79$ we get from (52) that $\beta(0.01)=2.77$, $\beta(0.05)=2.71$ and $\beta(0.1)=2.66$. These results resemble those derived earlier for material (2b) and displayed in Fig. 1.

The critical temperature difference θ^*, at the onset of a second plastic zone, is shown in Fig. 4. These curves represent the respective solutions of (50), and (51) with (34). Note that all material characteristics are lumped here into the single parameter H. When $H=0$ we recover again the solution of [1, 2] for an elastic/perfectly-plastic material.

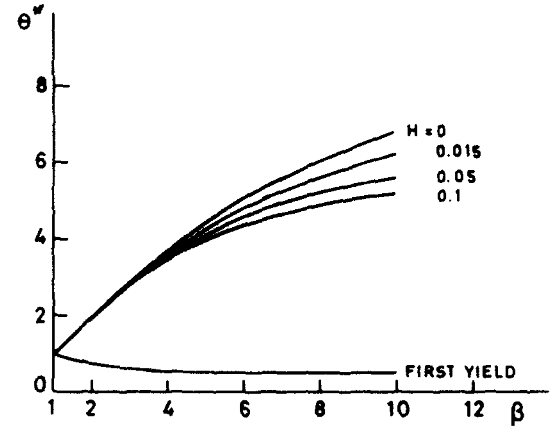

Fig. 4. Temperature difference required for the onset of a second plastic zone in an elastic/linear-hardening material.

Acknowledgements—Thanks are due to the kind support of the Lord-Marks Fellowship. The author would like also to acknowledge the kind assistance of the University Engineering Department, and the warm hospitality of Churchill College, Cambridge. Helpful discussions with Aharon Durban, B.Sc., are reflected within the paper.

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